TRANSMISSION LINE PARAMETERS - CAPACITANCES AND INDUCTANCES

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Overhead Lines

The method of images is applied to calculate the capacitances and inductances of an overhead line which is strung parallel to and above the ground plane, assuming the earth to be a perfect conductor. This is schematically shown in Fig. 1.

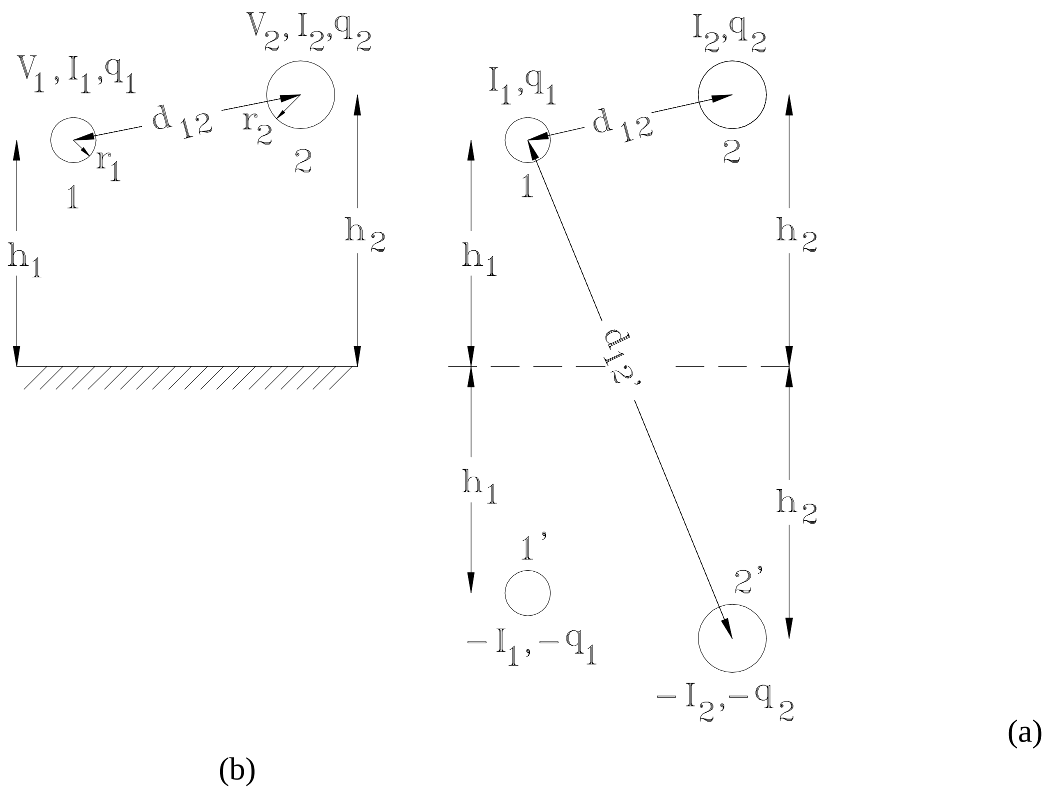

Fig. 1 Schematic of a two-conductor overhead line.

Figure 1(a) shows the schematic of a two-conductor line above a perfect earth. Fig. 1(b) is the equivalent circuit of the two-conductor line where the earth is replaced by the images of the two conductors placed below the earth's surface at distances equal to the heights (h_1, h_2) of the two respective conductors. The image conductors (1',2') have voltage, current and charge which are equal in magnitude but opposite in sign to that of the corresponding line conductors (1,2). The charges and the currents of the image conductors, along with those of the actual conductors (1,2), reproduce the electric and magnetic fields on the earth's surface of the actual configuration of Fig. 1(a).

Capacitance to earth of a conductor above earth

If q_1 (C/m) is the charge on conductor 1, then the electric field, E, at a distance, r, from the center of conductor 1 can be found by applying Gauss' law:

$$q_1 = 2\pi r \epsilon_o E, \quad \text{or,} \quad E = \frac{q_1}{2\pi r \epsilon_o}, \qquad \textbf{(1)}$$

where ϵ_o is the permittivity of air. Assuming the center of conductor 1 to be at the origin and integrating from the earth's surface, which is at zero potential, to the surface of conductor 1, the voltage (with respect to earth), V_{11}, is found as:

$$V_{11} = -\int_{h_1}^{r_1} E \cdot dr = -\frac{q_1}{2\pi\epsilon_o}\int_{h_1}^{r_1}\frac{dr}{r} = \frac{q_1}{2\pi\epsilon_o}\ell n\frac{h_1}{r_1}, \qquad \textbf{(2)}$$

where r_1 is the radius of conductor 1.

Similarly, let the voltage on conductor 1 due to the charge ($-q_1$ C/m) on the image conductor, 1', be $V_{11'}$. Again, invoking Gauss' law and shifting the origin to the center of the image conductor, 1':

$$V_{11'} = \frac{q_1}{2\pi\epsilon_o}\ell n\frac{2h_1 - r_1}{h_1}. \qquad \textbf{(3)}$$

As $2h_1 >> r_1$, $2h_1 - r_1 \approx 2h_1$. Adding (2) and (3), the total voltage, V_1, on conductor 1 is:

$$V_1 = V_{11} + V_{11'} = \frac{q_1}{2\pi\epsilon_o}\ell n\frac{2h_1}{r_1} = p_{11}q_1, \qquad \textbf{(4)}$$

$$\text{where,} \quad p_{11} = \frac{\ell n\frac{2h_1}{r_1}}{2\pi\epsilon_o}. \qquad \textbf{(5)}$$

p_{11} is called the potential coefficient. The total capacitance (F/m) of conductor 1to ground, C_{1g}, is:

$$C_{1g} = \frac{q_1}{V_1} = \frac{2\pi\epsilon_o}{\ell n\frac{2h_1}{r_1}} = \frac{1}{p_{11}}, \quad \text{(F/m)}. \qquad \textbf{(6)}$$

Mutual capacitance between two conductors above earth

Referring to Fig. 1(b), with a charge, q_2 (C/m), on conductor 2, the electric field, E, at a point, distant r from the center of conductor 2 and along the line 12 is:

$$E = \frac{q_2}{2\pi\epsilon_o r}. \qquad \textbf{(7)}$$

Integrating the electric field between conductors 1 and 2 from d_{12} to r_2:

$$V_{1-2} = -\frac{q_2}{2\pi\epsilon_o}\int_{r_2}^{d_{12}-r_1}\frac{dr}{r} = -\frac{q_2}{2\pi\epsilon_o}\ell n\frac{d_{12}-r_1}{r_2}\ . \qquad \textbf{(8)}$$

Similarly, the voltage between conductor 1 and the image conductor 2' is:

$$V_{1-2'} = \frac{q_2}{2\pi\epsilon_o}\ell n\frac{d_{12'}-r_1}{r_2}\ . \qquad \textbf{(9)}$$

Adding (8) and (9):

$$V_{12} = V_{1-2} + V_{1-2'} = \frac{q_2}{2\pi\epsilon_o}\ell n\frac{d_{12'}-r_1}{d_{12}-r_1} \approx \frac{q_2}{2\pi\epsilon_o}\ell n\frac{d_{12'}}{d_{12}} = p_{12}q_2\ , \qquad \textbf{(10)}$$

$$\text{where},\quad p_{12} = \frac{\ell n\dfrac{d_{12'}}{d_{12}}}{2\pi\epsilon_o}\ . \qquad \textbf{(11)}$$

p_{12} is the potential coefficient between conductors 1 and 2. The mutual capacitance between conductor 1 and conductor 2 is then:

$$C_{1-2} = \frac{q_2}{V_{12}} = \frac{2\pi\epsilon_o}{\ell n\dfrac{d_{12'}}{d_{12}}} = \frac{1}{p_{12}}\ ,\quad (\text{F/m}). \qquad \textbf{(12)}$$

Self inductance of a single conductor above earth

Referring to Fig. 1(b), it should be observed that each conductor above earth (1 and 2) forms a closed loop with its image conductor (1' and 2'). The inductance of each loop can be determined by invoking Ampere's law. For the circuit 1-1':

$$I_1 = \oint H_1 \cdot dr = H_1 \cdot 2\pi r, \quad \text{or}, \quad H_1 = \frac{I_1}{2\pi r}, \qquad \textbf{(13)}$$

where H_1 is the magnetic field surrounding conductor 1 at a radial distance, r, from the center of conductor 1 due to its own current, I_1. The magnetic induction (or flux density), $B_1 = \mu_o H_1$, where μ_o ($4\pi x 10^{-7}$) is the permeability of air. The total magnetic flux enclosed between conductor 1 and the earth's surface is:

$$\phi_1 = \int_{r_1}^{h_1 - r_1} B_1 \cdot dr = \frac{\mu_o I_1}{2\pi} \int_{r_1}^{h_1 - r_1} \frac{dr}{r} \approx \frac{\mu_o}{2\pi} \ell n \frac{h_1}{r_1} . \qquad \textbf{(14)}$$

Similarly, shifting the origin to the center of image conductor 1', the magnetic flux from the earth's surface to the surface of conductor 1 due to current, $-I_1$, flowing in the image conductor 1' is:

$$\phi_{1'} = \frac{\mu_o}{2\pi} \ell n \frac{2h_1 - r_1}{h_1} . \qquad \textbf{(15)}$$

The total magnetic flux, due to currents in conductors 1 and 1', is:

$$\phi_{11} = \phi_1 + \phi_{1'} = \frac{\mu_o I_1}{2\pi} \ell n \frac{2h_1 - r_1}{r_1} \approx \frac{\mu_o I_1}{2\pi} \ell n \frac{2h_1}{r_1} . \qquad \textbf{(16)}$$

The self inductance, L_{11}, being $L_{11}=\phi_{11}/I_1$,

$$L_{11} = \frac{\phi_{11}}{I_1} = \frac{\mu_o}{2\pi} \ell n \frac{2h_1}{r_1} = 2\times10^{-7} \ell n \frac{2h_1}{r_1}, \quad \text{(H/m)} . \qquad \textbf{(17)}$$

Similarly, for conductor 2:

$$L_{22} = 2\times10^{-7} \ell n \frac{2h_2}{r_2}, \quad \text{(H/m)}. \qquad \textbf{(18)}$$

Mutual inductance between two conductors above earth

Referring to Fig. 1(b), the magnetic flux between conductors 1 and 2, created by current, I_1, in conductor 1 is:

$$\phi_{12} = \frac{\mu_o I_1}{2\pi}\int_{d_{12}-r_2}^{r_1}\frac{dr}{r} = \frac{\mu_o I_1}{2\pi}\ell n\frac{r_1}{d_{12}-r_2} \approx \frac{\mu_o I_1}{2\pi}\ell n\frac{r_1}{d_{12}}, \tag{19}$$

Similarly, the magnetic flux between the image conductor 1' and conductor 2, due to current, $-I_1$, in the image conductor 1' is:

$$\phi_{1'2} = -\frac{\mu_o I_1}{2\pi}\ell n\frac{r_1}{d_{1'2}} = \frac{\mu_o I_1}{2\pi}\ell n\frac{d_{12'}}{r_1} \ . \tag{20}$$

The total magnetic flux between conductor 2 and the earth's surface, created by currents in conductor 1 and its image conductor 1' is:

$$\phi_2 = \phi_{12} + \phi_{1'2} = \frac{\mu_o I_1}{2\pi}\ell n\frac{d_{12'}}{d_{12}} \ . \tag{21}$$

$$L_{12} = \frac{\phi_2}{I_1} = \frac{\mu_o}{2\pi}\ell n\frac{d_{12'}}{d_{12}} = 2\times10^{-7}\ell n\frac{d_{12'}}{d_{12}}, \quad (H/m) \ . \tag{22}$$

Multiconductor overhead lines

Invoking (4) and (10), the total voltage of each conductor of a two-conductor overhead line can be written as:

$$V_1 = p_{11}q_1 + p_{12}q_2, \quad \text{and} \quad V_2 = p_{21}q_1 + p_{22}q_2 \ . \qquad \textbf{(23)}$$

Extending to an n-conductor system:

$$V_1 = p_{11}q_1 + p_{12}q_2 + ... + p_{1r}q_r + ... + p_{1n}q_n \ , \qquad \textbf{(24)}$$

$$V_2 = p_{21}q_1 + p_{22}q_2 + ... + p_{2r}q_r + ... + p_{2n}q_n \ , \qquad \textbf{(25)}$$

$$\vdots$$

$$V_r = p_{1r}q_1 + p_{2r}q_2 + ... + p_{rr}q_r + ... + p_{rn}q_n \ , \qquad \textbf{(26)}$$

$$\vdots$$

$$V_n = p_{n1}q_1 + p_{n2}q_2 + ... + p_{nr}q_r + ... + p_{nn}q_n \ . \qquad \textbf{(27)}$$

In matrix form:

$$\begin{bmatrix} V_1 \\ V_2 \\ . \\ . \\ V_r \\ . \\ V_n \end{bmatrix} = [V] = \begin{bmatrix} p_{11} & p_{12} & . & . & . & p_{1r} & . & . & . & p_{1n} \\ p_{21} & p_{21} & . & . & . & p_{2r} & . & . & . & p_{2n} \\ . & & & & & & & & & \\ p_{r1} & p_{r2} & . & . & . & p_{rr} & . & . & . & p_{rn} \\ . & & & & & & & & & \\ p_{n1} & p_{n2} & . & . & . & p_{rn} & . & . & . & p_{nn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ . \\ q_r \\ . \\ q_n \end{bmatrix} = [p][q] \ , \qquad \textbf{(28)}$$

$$\text{where,} \quad p_{rr} = \frac{1}{2\pi\epsilon_o} \ell n \frac{2h_r}{r_r}, \quad \text{and} \quad p_{rs} = \frac{1}{2\pi\epsilon_o} \ell n \frac{d_{rs^/}}{d_{rs}} . \qquad \textbf{(29)}$$

$$\text{From (28),} \quad [q] = [p]^{-1}[V] = [C][V], \quad \text{where} \quad [C] = [p]^{-1} . \qquad \textbf{(30)}$$

Expanding (30):

$$q_1 = C_{11}V_1 + C_{12}V_2 + ... + C_{1r}V_r + ... + C_{1n}V_n, \quad \textbf{(31)}$$

$$q_2 = C_{21}V_1 + C_{22}V_2 + ... + C_{2r}V_r + ... + C_{2n}V_n, \quad \textbf{(32)}$$

$$\dot{q}_r = C_{r1}V_1 + C_{r2}V_2 + ... + C_{rr}V_r + ... + C_{rn}V_n, \quad \textbf{(33)}$$

$$\dot{q}_n = C_{n1}V_1 + C_{n2}V_2 + ... + C_{nr}V_r + ... + C_{nn}V_n. \quad \textbf{(34)}$$

The diagonal terms, C_{rr} of the matrix [C] are called the coefficients of capacitance, and the off-diagonal terms, C_{rs}, are called the coefficients of induction.

A three-conductor system above earth with its equivalent capacitive network is shown in Fig. 2.

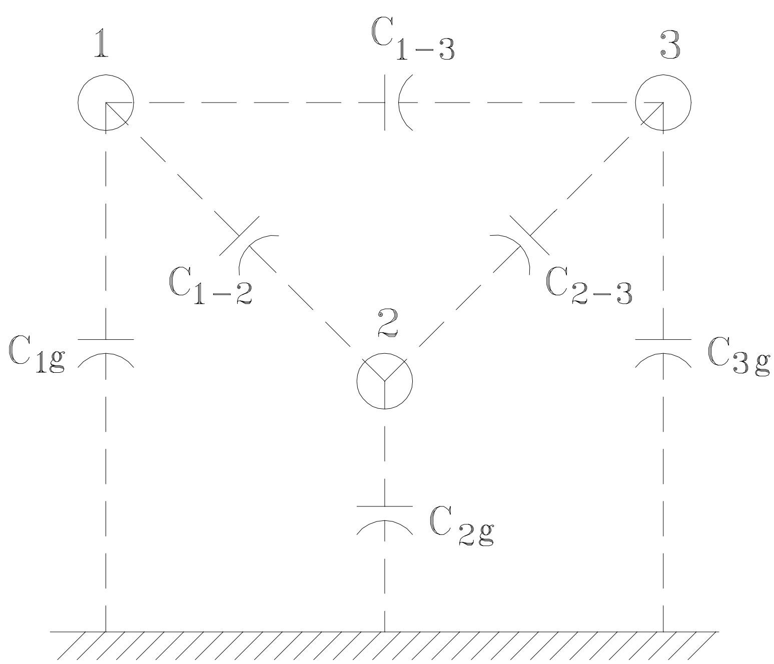

Fig. 2 Multiconductor system with capacitive coupling above earth.

If the voltages and charges (C/m) of the three-conductor system above earth, shown in Fig. 2, are V_1, V_2, V_3 and q_1 , q_2 , q_3 , then extending to a general n-conductor system, the conductor charges are:

$$q_1 = C_{1g}V_1 + C_{1-2}(V_1 - V_2) + C_{1-3}(V_1 - V_3), \quad \textbf{(35)}$$

$$q_2 = C_{1-2}(V_2 - V_1) + C_{2g}V_2 + C_{2-3}(V_2 - V_3), \quad \textbf{(36)}$$

$$q_3 = C_{1-3}(V_1 - V_3) + C_{2-3}(V_2 - V_3) + C_{3g}V_3. \quad \textbf{(37)}$$

Rearranging (35)-(37):

$$q_1 = (C_{1g} + C_{1-2} + C_{1-3})V_1 - C_{1-2}V_2 - C_{1-3}V_3, \quad \textbf{(38)}$$

$$q_2 = -C_{1-2}V_1 + (C_{2g} + C_{1-2} + C_{1-3})V_2 - C_{2-3}V_3, \quad \textbf{(39)}$$

$$q_3 = -C_{1-3}V_1 - C_{2-3}V_2 + (C_{3g} + C_{1-3} + C_{2-3})V_3. \quad \textbf{(40)}$$

Equations (38) to (40) will be the same as (31) to (34) by putting:

$C_{11}=C_{1g}+C_{1-2}+C_{1-3}$; $C_{22}=C_{2g}+C_{1-2}+C_{2-3}$; $C_{33}=C_{3g}+C_{1-3}+C_{2-3}$; and
$C_{12}= - C_{1-2}$; $C_{13}= - C_{1-3}$; $C_{23}= - C_{2-3}$.

Bearing in mind that C_{1g}, C_{2g}, C_{3g},..., C_{ng}, are the real capacitances of the conductors to ground and C_{1-2}, C_{1-3}, C_{2-3} ,...C_{m-n} are mutual capacitances between conductors, these capacitances can be determined once the capacitance matrix, $[C]=[p]^{-1}$, is known:

$$C_{rg} = \sum_{s=1}^{n} C_{rs} \;, \quad \text{and} \quad C_{r-s} = -C_{rs}. \quad \textbf{(41)}$$

Simply put, the capacitance of the r-th conductor to ground is the sum of all the elements of the r-th row or the r-th column of the capacitance matrix.

Underground Cables

The cross-section of a coaxial cable is shown in Fig. 3 where the inner conductor, of outer radius a, is separated from the outer conductor, of inner radius b, by a dielectric material of dielectric constant, k.

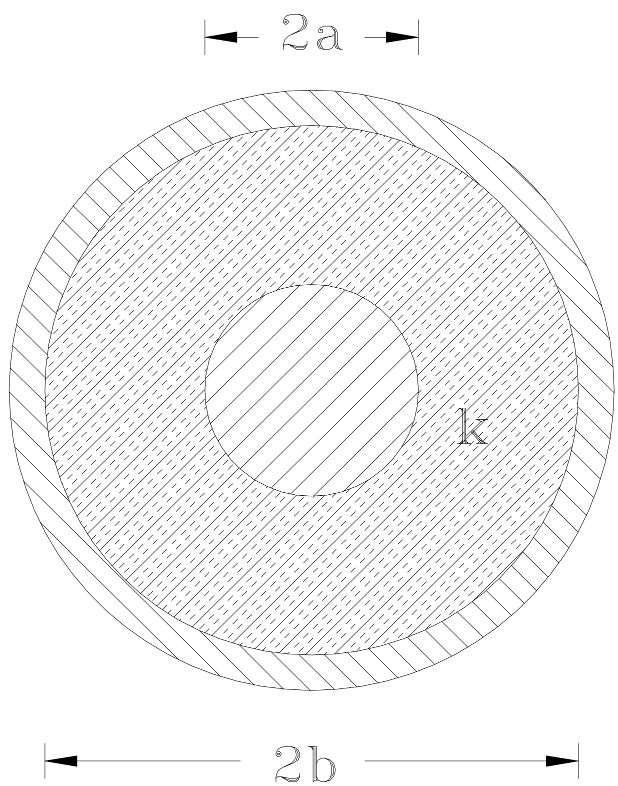

Fig. 3 Cross-section of a coaxial cable.

Capacitance

If the charge of the inner conductor is q (C/m), then the electric field at a radial distance r from the center of the cable is, by invoking Gauss' law:

$$E = \frac{q}{2\pi\epsilon_o k r} , \qquad \textbf{(42)}$$

$$V = -\frac{q}{2\pi\epsilon_o k}\int_b^a \frac{dr}{r} = \frac{q}{2\pi\epsilon_o}\ell n\frac{b}{a} , \quad \text{and} \qquad \textbf{(43)}$$

$$C = \frac{q}{V} = \frac{2\pi\epsilon_o k}{\ell n \frac{b}{a}}, \quad (\mathrm{F/m}). \qquad \textbf{(44)}$$

Inductance

If the inner conductor carries a current, I, then applying Ampere's law, the magnetic field at a distance r from the center of the inner conductor is:

$$H = \frac{I}{2\pi r}, \qquad \textbf{(45)}$$

The magnetic flux enclosed in the space between the two conductors is:

$$\phi = \frac{\mu_o I}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_o I}{2\pi}\ell n\frac{b}{a}, \quad \text{and} \qquad \textbf{(46)}$$

$$L = \frac{\phi}{I} = \frac{\mu_o}{2\pi}\ell n\frac{b}{a} = 2\times 10^{-7}\ell n\frac{b}{a}, \quad (\mathrm{H/m}). \qquad \textbf{(47)}$$